

SOME TRIGONOMETRIC POLYNOMIALS WITH EXTREMELY SMALL UNIFORM NORM

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ABSTRACT. An example of trigonometric polynomials with extremely small uniform norm is given. This example demonstrates the potential limits for extension of Sidon's inequality for lacunary polynomials in a certain direction.

Key words: Sidon's inequality, lacunary polynomials.

We prove the following

Theorem 1. *Let $q > 1$ and a sequence of naturals $\{m_j\}_{j=1}^\infty$ satisfy $m_{j+1}/m_j \geq q$ for all j . Let another sequence of naturals $\{d_j\}_{j=1}^\infty$ satisfy $1 \leq d_j \leq m_{j+1} - m_j$. Then there exists a sequence of trigonometric polynomials $\{\delta_j\}_{j=1}^\infty$ such that*

$$\delta_j(x) = \sum_{m_j \leq s < m_j + d_j} c_s e^{isx}, \quad (1)$$

$$\frac{1}{8} \leq \|\delta_j\|_1 \leq \|\delta_j\|_\infty \leq 7, \quad (2)$$

$$\left\| \sum_{j=1}^N \delta_j \right\|_\infty \leq \alpha + \beta \sqrt{N} + \gamma \max_{1 \leq j \leq N} \log_q \max \left(\frac{m_j}{d_j}, \frac{1}{\ln q}, 1 \right) \quad (3)$$

for all $N = 1, 2, \dots$ with some positive absolute constants α, β and γ .

This result improves the examples constructed by Grigoriev [1] and Radomskii [6], where roughly speaking Theorem 1 was proved for the case $m_j = 2^j$, $d_j = \lfloor 2^{j-j^\varepsilon} \rfloor$ and some small limitations.

Remark 1. In Theorem 1 the example is constructed with the constants $\alpha = 316$, $\beta = 7\sqrt{2c_H}$, $\gamma = 210$, where c_H is the constant from the Carleson-Hunt inequality, see (5) below. Instead of the Carleson-Hunt result one could use a well-known simpler inequality (see [7], Ch. 13, Th. 1.17)

$$\left\| \sup_{1 \leq j < \infty} \left| \sum_{k=1}^{m_j} b_k e^{ikx} \right| \right\|_2^2 \leq A_q \sum_{k=1}^{\infty} |b_k|^2$$

whenever $m_{j+1}/m_j \geq q$ with a constant A_q depending on q . The Carleson-Hunt inequality is used here because we would like to make the constants α, β

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and γ independent of q . Otherwise we do not try to optimize the constants in Theorem 1, e.g. for the case $q = 2$ it is not difficult to repeat the arguments of our proof and get some essentially better constants.

Remark 2. In this paper we assume that the norms of $L_p(0, 2\pi)$ are normalized, i.e. $\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f|^p d\mu\right)^{1/p}$ for $1 \leq p < \infty$, where μ is the standard Lebesgue measure so that $\|1\|_p = 1$.

Remark 3. One can deduce as a corollary a version of Theorem 1 with *real* polynomials δ_j with slight changes in the constants. (We decided to prove the result for δ_j with *positive frequencies* and *real coefficients*.) Also with minor changes in the proof one can prove a version of Theorem 1 with $\delta_j(x) = p_j(x) \cos m_j x$ with some real trigonometric polynomials p_j such that $\deg p_j \leq d_j$.

Taking Remark 2 into account we derive from Theorem 1 the following

Corollary 1. *Let $0 \leq \varepsilon < 1$. Then there exists a sequence of real trigonometric polynomials $\{p_j\}_{j=1}^\infty$ such that $\deg p_j \leq 2^{j-j^\varepsilon}$, $c_- \leq \|p_j\|_1 \leq \|p_+\|_\infty \leq c_+$ and*

$$\left\| \sum_{j=1}^N p_j(x) \cos 2^j x \right\|_\infty \leq C N^{\max(\varepsilon, \frac{1}{2})}$$

for all $N = 1, 2, \dots$ with some positive absolute constants c_- , c_+ , C .

The example constructed in Theorem 1 gives some limits for the attempts to extend the well-known property of lacunary polynomials (Sidon's inequality)

$$\left\| \sum_{j=1}^N a_j \cos m_j x \right\|_\infty \geq c(q) \sum_{j=1}^N |a_j|$$

with some $c(q) > 0$ whenever $m_{j+1}/m_j \geq q > 1$. The question of possible extension of Sidon's inequality with substituting $a_j \cos m_j x$ by $p_j(x) \cos m_j x$ with p_j being trigonometric polynomials of possibly large degree was raised by Kashin and Temlyakov [4] and Radomskii [5] in association with estimating the entropy numbers of certain functional classes.

The scheme of proof follows that from Grigoriev [1], [2] and Radomskii [6] with minor changes. This technique invented in [1] could be called the *pseudo stopping time method* since the idea was borrowed from stochastic analysis (see [2] for explanations).

PROOF OF THEOREM 1. It is easy to notice that without loss of generality we can consider only the case when

$$\frac{m_j}{d_j} \geq \max \left(1, \frac{1}{\ln q} \right). \quad (4)$$

By the famous result of Carleson and Hunt [3] the L_2 -norm of the majorant of a trigonometric sum can be estimated as

$$\left\| \max_{0 \leq k \leq n} \left| \sum_{s=0}^k b_s e^{isx} \right| \right\|_2^2 \leq c_H \sum_{s=1}^n |b_s|^2 \quad (5)$$

with an absolute constant $c_H > 0$.

We construct the polynomials δ_{k_j} by induction with the constants

$$\alpha := 316, \quad \beta := 7\sqrt{2c_H}, \quad \gamma := 210. \quad (6)$$

Let $\delta_1(x) := \exp(im_1 x)$. On each inductive step in order to construct δ_n having $\delta_1, \dots, \delta_{n-1}$ from the previous steps we denote

$$S_j := \sum_{k=1}^j \delta_k$$

and define

$$E_n^j := \{x \in [0, 2\pi) : |S_j(x)| > \beta\sqrt{n}\}; \quad (7)$$

$$B_n := \bigcup_{j=1}^{n-1} E_n^j; \quad (8)$$

$$\tilde{B}_n := \bigcup_{1 \leq j \leq n-a_n} O_{2\pi \frac{(n-k)^2}{d_n}}(E_n^j), \quad (9)$$

$$\text{where } a_n := 45 + 30 \log_q \frac{m_n}{d_n} \quad (10)$$

and $O_\varepsilon(X)$ denotes the ε -neighborhood of the set X on the semi-interval $[0, 2\pi)$ with the circle metric. *Note that the value $n - a_n$ need not to be neither integer nor positive. If $n - a_n < 1$, the union in (9) is understood as an empty set. Similarly below the sums like $\sum_{1 \leq j \leq n-a_n}$ are supposed equal to zero whenever $n - a_n < 1$.*

Set

$$\Lambda_n := \left\{ l \in \{1, \dots, d_n\} : 2\pi \frac{l}{d_n} \notin \tilde{B}_n \right\}; \quad (11)$$

$$\delta_n(x) := \frac{1}{d_n} \exp \left\{ i \left(m_n + \left[\frac{d_n - 1}{2} \right] x \right) \right\} \sum_{l \in \Lambda_n} K_{\left[\frac{d_n - 1}{2} \right]} \left(x - 2\pi \frac{l}{d_n} \right), \quad (12)$$

where $K_d(x) = \frac{2}{d+1} \left(\frac{\sin \frac{1}{2}(d+1)x}{2 \sin \frac{x}{2}} \right)^2$ are the Fejér kernels. Clearly so defined δ_n satisfies the frequency constraint imposed by (1).

In order to verify the induction hypothesis we need to show that

$$|\Lambda_n| > \frac{1}{4} d_n. \quad (13)$$

By the induction hypothesis we have (2) for $\delta_1, \dots, \delta_{n-1}$. Applying the Chebyshev inequality for the majorant $S_{n-1}^* := \max_{1 \leq j \leq n-1} \left| \sum_{s=1}^j \delta_s(x) \right|$ (see (7), (8)) and then using (5), (2) and (6) we get

$$\mu B_n \leq \frac{2\pi \|S_{n-1}^*\|_2^2}{\beta^2 n} \leq \frac{2\pi c_H \|S_{n-1}\|_2^2}{\beta^2 n} \leq \frac{2\pi}{\beta^2 n} \sum_{j=1}^{n-1} \|\delta_{k_j}\|_\infty^2 \leq \frac{2\pi}{\beta^2 n} (n-1)7^2 < \pi. \quad (14)$$

Let us denote by $\text{Conn}(X)$ the number of connected components of a set $X \subset [0, 2\pi)$ in the circle topology. Note that $|S_j|^2$ is a real trigonometric polynomial of degree not exceeding $2(m_j + 2[\frac{d_j-1}{2}])$ and therefore the equation $|S_j(x)|^2 = \beta^2 n$ has not more than $4(m_j + 2[\frac{d_j-1}{2}])$ roots. Note that E_n^j is a finite union of open intervals which endpoints are the roots of $|S_j(x)|^2 = \beta^2 n$. Taking into account (4) we conclude

$$\text{Conn}(E_n^j) \leq \frac{1}{2} |\{x : |S_j(x)|^2 = \beta^2 n\}| \leq 2 \left(m_j + 2 \left[\frac{d_j-1}{2} \right] \right) < 4m_j. \quad (15)$$

This implies

$$\begin{aligned} \text{Conn}(\tilde{B}_n) &\leq \sum_{1 \leq j \leq n-a_n} \text{Conn}(E_n^j) < 4 \sum_{1 \leq j \leq n-a_n} m_j \leq 4 \sum_{1 \leq j \leq n-a_n} m_n q^{j-n} \\ &< 4m_n \sum_{s \geq a_n} q^{-s} = 4m_n q^{-[a_n]} \frac{q}{q-1}, \end{aligned}$$

where $[x]$ denotes the ceiling integer part of x . Taking into account that $a_n \leq [a_n]$, $\ln q \leq q-1$ and using (10) and (4) we proceed as

$$\begin{aligned} \text{Conn}(\tilde{B}_n) &< 4m_n q^{-a_n} \frac{q}{q-1} = 4m_n q^{-45} \left(\frac{m_n}{d_n} \right)^{-30} \frac{q}{q-1} = 4d_n \left(\frac{m_n}{d_n} \right)^{-29} \frac{q^{-44}}{q-1} \\ &\leq 4d_n \left(\frac{m_n}{d_n} \right)^{-29} \frac{q^{-44}}{\ln q} \leq 4d_n \left(\frac{m_n}{d_n} \right)^{-28} q^{-44} \leq 4d_n \max \left(\left(\frac{m_n}{d_n} \right)^{-28}, q^{-44} \right). \end{aligned}$$

If $q \geq e^{1/2}$, then $q^{-44} \leq e^{-22}$. If $q \leq e^{1/2}$, then $\left(\frac{m_n}{d_n} \right)^{-28} \leq \left(\frac{1}{\ln q} \right)^{-28} \leq 2^{-28}$. So we get

$$\text{Conn}(\tilde{B}_n) < 4d_n \max \left(2^{-28}, e^{-22} \right) < \frac{d_n}{8}. \quad (16)$$

Aggregating (14) and (15) (see also (9)) we get

$$\begin{aligned} \mu \tilde{B}_n &\leq \mu B_n + 2 \sum_{1 \leq j \leq n-a_n} 2\pi \frac{(n-j)^2}{d_n} \text{Conn}(E_n^j) \leq \pi + 16\pi \sum_{1 \leq j \leq n-a_n} \frac{(n-j)^2}{d_n} m_j \\ &\leq \pi + 16\pi \sum_{1 \leq j \leq n-a_n} \frac{(n-j)^2}{d_n} m_n q^{j-n} = \pi + 16\pi \frac{m_n}{d_n} \sum_{a_n \leq s \leq n-1} s^2 q^{-s}. \end{aligned}$$

To proceed with the estimate of $\mu\tilde{B}_n$ we are going to use the inequality

$$\sum_{s=a}^{\infty} s^2 q^{-s} \leq \frac{2a^2 q^{3-a}}{(q-1)^3} \quad \text{for } a = 1, 2, \dots$$

This inequality one could easily deduce from the following identity

$$\sum_{s=a}^{\infty} s^2 q^{-s} = \frac{q^{3-a}}{(q-1)^3} \left(a^2 + q^{-1}(1+2a-2a^2) + q^{-2}(a-1)^2 \right).$$

for all $q > 1$ and $a = 0, 1, \dots$, which is not so difficult to verify.

Now proceed with the estimate of $\mu\tilde{B}_n$ as

$$\mu\tilde{B}_n < \pi + 32\pi \frac{m_n}{d_n} \frac{[a_n]^2 q^{3-[a_n]}}{(q-1)^3} < \pi + 32\pi \frac{m_n}{d_n} \frac{[a_n]^2 q^{3-[a_n]}}{\ln^3 q}.$$

The function $x^2 q^{-x}$ is decreasing for $x \geq 2/\ln q$. One easily checks that (4) and (10) imply $a_n \geq 2/\ln q$ and therefore we can substitute $[a_n]$ for a_n in the right-hand side above. So recalling (10) and (4) again we continue as

$$\begin{aligned} \mu\tilde{B}_n &< \pi + 32\pi \frac{m_n}{d_n} \frac{a_n^2 q^{3-a_n}}{\ln^3 q} \\ &= \pi + 32\pi \left(\frac{m_n}{d_n} \right)^{-29} \frac{(45 + 30 \log_q \frac{m_n}{d_n})^2 q^{-42}}{\ln^3 q} \\ &\leq \pi + 32\pi \left(\frac{m_n}{d_n} \right)^{-26} (45 + 30 \log_q \frac{m_n}{d_n})^2 q^{-42} \\ &\leq \pi + 64\pi \left(\frac{m_n}{d_n} \right)^{-26} \left\{ 45^2 + 30^2 \left(\frac{\ln \frac{m_n}{d_n}}{\ln q} \right)^2 \right\} q^{-42} \\ &\leq \pi + 64\pi \left\{ 45^2 \left(\frac{m_n}{d_n} \right)^{-26} + 30^2 \left(\frac{m_n}{d_n} \right)^{-22} \right\} q^{-42}. \end{aligned}$$

If $q \geq e^{1/2}$, then

$$\left\{ 45^2 \left(\frac{m_n}{d_n} \right)^{-26} + 30^2 \left(\frac{m_n}{d_n} \right)^{-22} \right\} q^{-42} \leq (45^2 + 30^2) e^{-21} < (64^2 + 64^2) 2^{-21} = 2^{-8}.$$

If $q \leq e^{1/2}$, then $\frac{m_n}{d_n} \geq 1/\ln q \geq 2$ and

$$\left\{ 45^2 \left(\frac{m_n}{d_n} \right)^{-26} + 30^2 \left(\frac{m_n}{d_n} \right)^{-22} \right\} q^{-42} < 45^2 2^{-26} + 30^2 2^{-22} < 2^{-11}.$$

So we finally conclude

$$\mu\tilde{B}_n < \pi + 64\pi 2^{-8} = \frac{5}{4}\pi. \quad (17)$$

Now we are ready to prove (13). Clearly, the number of elements in Λ_n^c does not exceed the number of the intervals of type $(\frac{2\pi l - \pi}{d_n}, \frac{2\pi l + \pi}{d_n})$ ($l = 1, \dots, d_n$) which intersect \tilde{B}_n (see (11)). Such intervals can be split into two groups: those

containing an edge point of \tilde{B}_n and those being included in \tilde{B}_n . There are not more than $2\text{Conn}(\tilde{B}_n) \leq d_n/8$ of the intervals of the first type (see (16)). Denoting by V the number of the intervals of the second type and recalling (17) we get

$$V \frac{2\pi}{d_n} \equiv V \mu\left(-\frac{\pi}{d_n}, \frac{\pi}{d_n}\right) \leq \mu \tilde{B}_n < \frac{5}{4}\pi.$$

So we have

$$|\Lambda_n| \geq d_n - V - \frac{d_n}{8} > d_n - \frac{5}{8}d_n - \frac{1}{8}d_n = \frac{1}{4}d_n.$$

Thus we proved (13).

Our next goal is to verify (2) for δ_n . Let us recall some properties of the Fejér kernels. It is well-known that for all $-\pi \leq x \leq \pi$ and $d = 1, 2, \dots$

$$K_{d-1}(x) \geq 0, \quad \|K_{d-1}\|_1 = 1/2, \quad (18)$$

$$K_{d-1}(x) \leq \min\left(\frac{d}{2}, \frac{\pi^2}{2dx^2}\right) < 5 \min\left(d, \frac{1}{dx^2}\right). \quad (19)$$

Denote by $\text{dist}(x, y)$ the standard distance on the circle between $x, y \in [0, 2\pi)$ and let $\tilde{d}_n := \left[\frac{d_n-1}{2}\right] + 1$. Clearly, $1 \leq d_n/\tilde{d}_n \leq 2$. Using (12), (13), (18) and (19) we verify (2) as follows

$$\|\delta_n\|_1 = \frac{1}{d_n} \left\| \sum_{l \in \Lambda_n} K_{\tilde{d}_n-1}\left(x - 2\pi \frac{l}{d_n}\right) \right\|_1 = \frac{|\Lambda_n|}{d_n} \|K_{\tilde{d}_n-1}\|_1 \geq \frac{1}{8}$$

and

$$\begin{aligned} \|\delta_n\|_\infty &= \frac{1}{d_n} \left\| \sum_{l \in \Lambda_n} K_{\tilde{d}_n-1}\left(x - 2\pi \frac{l}{d_n}\right) \right\|_\infty \\ &\leq \frac{1}{d_n} \left\| \sum_{l=1}^{d_n} 5 \min\left(\tilde{d}_n, \frac{1}{\tilde{d}_n \text{dist}(x, 2\pi \frac{l}{d_n})^2}\right) \right\|_\infty \\ &\leq 5 \left(1 + 2 \sum_{s=1}^{\infty} \frac{1}{\tilde{d}_n^2 (2\pi \frac{s}{d_n})^2}\right) \\ &= 5 \left(1 + \frac{d_n^2}{2\pi^2 \tilde{d}_n^2} \sum_{s=1}^{\infty} \frac{1}{s^2}\right) \\ &= 5 \left(1 + \frac{d_n^2}{2\pi^2 \tilde{d}_n^2} \frac{\pi^2}{6}\right) < 7. \end{aligned}$$

Now to complete the proof it remains to verify (3) with the constants (6), i.e. to show that

$$|S_n(x)| \leq \alpha + \beta\sqrt{n} + \gamma \max_{1 \leq j \leq n} \log_q \max\left(\frac{m_j}{d_j}, \frac{1}{\ln q}, 1\right) \quad \text{for each } x \in [0, 2\pi).$$

Set

$$\tau(x) := \max\{t = 1, \dots, n-1 : |S_t(x)| \leq \beta\sqrt{n}\}.$$

If $\tau(x) \geq n - a_n$, then using (2) and (10) we get

$$\begin{aligned} |S_n(x)| &\leq |S_{\tau(x)}(x)| + \sum_{t=\tau(x)+1}^n |\delta_t(x)| \leq \beta\sqrt{n} + 7a_n \\ &= \beta\sqrt{n} + 7\left(45 + 30 \log_q \frac{m_n}{d_n}\right) = \alpha - 1 + \beta\sqrt{n} + \gamma \log_q \frac{m_n}{d_n}. \end{aligned}$$

If $\tau(x) < n - a_n$, then

$$\begin{aligned} |S_n(x)| &\leq |S_{\tau(x)}(x)| + \sum_{t=\tau(x)+1}^{\tau(x)+a_n} |\delta_t(x)| + \sum_{t=\tau(x)+a_n+1}^n |\delta_t(x)| \\ &\leq \alpha - 1 + \beta\sqrt{n} + \gamma \log_q \frac{m_n}{d_n} + \sum_{t=\tau(x)+a_n+1}^n |\delta_t(x)|. \end{aligned} \quad (20)$$

It remains to estimate the last term in (20). Since $\tau(x) + a_n + 1 \leq t \leq n$, we have $x \in E_n^{\tau(x)+1} \subset E_t^{\tau(x)+1}$ (see (7)). Therefore, by the definition of \tilde{B}_t (see (9)) we conclude

$$\inf_{y \in [0, 2\pi) \setminus \tilde{B}_t} \text{dist}(x, y) \geq \frac{2\pi}{d_t} |t - \tau(x) - 1|^2.$$

Consequently, for each $l \in \Lambda_t$ we have (see (11))

$$\text{dist}\left(x, 2\pi \frac{l}{d_t}\right) \geq \frac{2\pi}{d_t} |t - \tau(x) - 1|^2.$$

Using (19) again and applying the trivial estimates $\sum_{s=K+1}^{\infty} s^{-2} < K^{-1}$ and $\sum_{s=K}^{\infty} s^{-2} < 2K^{-1}$ we conclude

$$\begin{aligned} |\delta_t(x)| &\leq \frac{1}{d_t} \sum_{l \in \Lambda_t} K_{\tilde{d}_t-1}\left(x - 2\pi \frac{l}{d_n}\right) \leq \frac{5}{d_t} \sum_{l \in \Lambda_t} \min\left(\tilde{d}_t, \frac{1}{\tilde{d}_t \text{dist}(x, 2\pi \frac{l}{d_t})^2}\right) \\ &\leq 10 \sum_{s=|t-\tau(x)-1|^2}^{\infty} \frac{1}{\tilde{d}_t d_t (2\pi \frac{s}{d_t})^2} = \frac{5d_t}{2\pi^2 \tilde{d}_t} \sum_{s=|t-\tau(x)-1|^2}^{\infty} \frac{1}{s^2} \\ &< \sum_{s=|t-\tau(x)-1|^2}^{\infty} \frac{1}{s^2} < \frac{2}{|t - \tau(x) - 1|^2}. \end{aligned}$$

Now we can estimate the last term in (20) as

$$\sum_{t=\tau(x)+a_n+1}^n |\delta_t(x)| < \sum_{t=\tau(x)+a_n+1}^n \frac{2}{|t - \tau(x) - 1|^2} = \sum_{s=a_n}^{\infty} \frac{2}{s^2} < \frac{2}{a_n - 1} < 1.$$

Using the last estimate in (20) we get (3) for S_n . This completes the proof.

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